From the 2019 Calculus Bee

① $\int \cos^2(5x)\,dx$

② Find $f'(2019)$ if

$f(x) = (x-1)(x-2)(x-3)\cdots(x-2019)$.

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① $\int \left(\frac{1}{2} + \frac{1}{2}\cos(10x)\right)dx = \frac{x}{2} + \frac{1}{20}\sin(10x) + C$.

② $f'(x) = (x-2)(x-3)\cdots(x-2019) + (x-1)(x-3)(x-4)+\cdots(x-2019)$

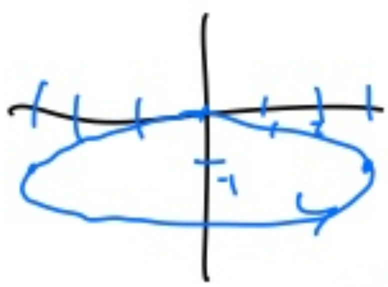
$\cdots + (x-1)(x-2)\cdots(x-2018)$

$f'(2019) = (2019-1)(2019-2)\cdots(2019-2018) = \boxed{2018!}$.

3.28 $I = \int_C x^2\,dy + (x-y)\,dx$ C = ellipse

$x^2 + 9y^2 + 18y = 0$

CCW



$x^2 + 9(y^2 + 2y + 1) = +9$

$x^2 + 9(y+1)^2 = 9$

$\frac{x^2}{9} + (y+1)^2 = 1$

parametrize

$x = 3\cos t$

$y = -1 + \sin(t)$

$$I = \int_0^{2\pi} 9\cos^2(t)\underbrace{(\cos(t)\,dt)}_{dy} + (3\cos(t) + 1 - \sin(t))(-3\sin(t)\,dt)$$

$$= \int_0^{2\pi} \underbrace{9\cos^3(t)}_{0} - \underbrace{9\cos(t)\sin(t)}_{0} - \underbrace{3\sin(t)}_{0} + 3\sin^2(t)\,dt$$

$$= 3 \cdot \frac{1}{2} \cdot 2\pi = \boxed{3\pi}$$

Green's Thm

$$I = \iint_{\text{inside}} d\left(x^2\,dy + (x-y)\,dx\right)$$

$$= \iint_{\text{inside}} (2x\,dx)\wedge dy + (\underbrace{dx}_{0} - dy)\wedge dx$$

$-dy\wedge dx = dx\wedge dy$

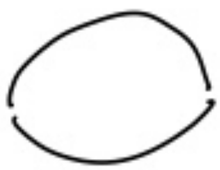
$$= \iint_{\text{ellipse}} (2x+1)\,dx\,dy$$

0

$$= \iint_{\text{ellipse}} 1\,dx\,dy = \text{area(ellipse)}$$

$$= \pi\cdot 3\cdot 1 = \boxed{3\pi}.$$

$\frac{x^2}{9}+(y+1)^2=1$



$(y+1)^2 = 1-\frac{x^2}{9}$

$y+1 = \pm\sqrt{1-\frac{x^2}{9}}$

$y = -1 \pm \sqrt{1-\frac{x^2}{9}}$

$$= \int_{-3}^{3}\int_{-1-\sqrt{1-\frac{x^2}{9}}}^{-1+\sqrt{1-\frac{x^2}{9}}} dy\,dx = \cdots = \boxed{3\pi}.$$

Variable substitution: $u=\frac{1}{3}x$ $\qquad u^2+v^2=1$

$v = y+1$

$du = \frac{1}{3}dx \qquad du\wedge dv = \frac{1}{3}dx\wedge dy$

$dv = dy \qquad 3\,du\wedge dv = dx\wedge dy.$

$$= \iint_{\substack{\text{unit}\\ \text{disk}}} 1\cdot 3\,du\wedge dv = 3\,\text{area}\left(\substack{\text{unit}\\ \text{disk}}\right)$$

$$= 3\pi$$

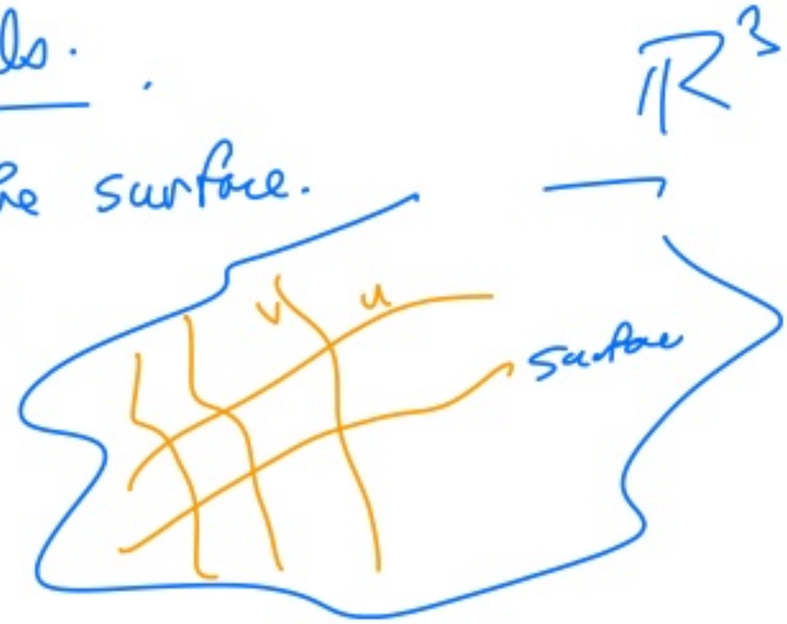
(u, v in polar) $\rightarrow \displaystyle\int_{\theta=0}^{2\pi}\int_{r=0}^{1} 3\cdot r\,dr\,d\theta$

Surface Integrals.

① Parametrize the surface.

$(x(u,v), y(u,v), z(u,v))$

need 2 parameters.



Examples

① Surface $z = x^3 y - y^2$

for $0 \le x \le 2$
$-1 \le y \le 4$

works any time $z=f(x,y)$

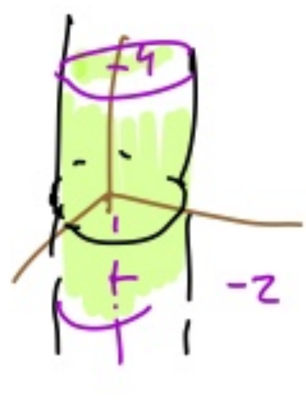
Let $u = x$, $v = y$

surface $(u, v, u^3 v - v^2)$ parametrization

$(x,y,z) = \Phi(u, v, u^3 v - v^2)$ $\quad 0 \le u \le 2$, $-1 \le v \le 4$

② Cylinder $x^2+y^2=1$

$-2 \le z \le 4$



$\Phi(u,v) = (\cos(u), \sin(u), v)$

$0 \le u \le 2\pi$
$-2 \le v \le 4$

③ Sphere of radius 4

use spherical coordinates

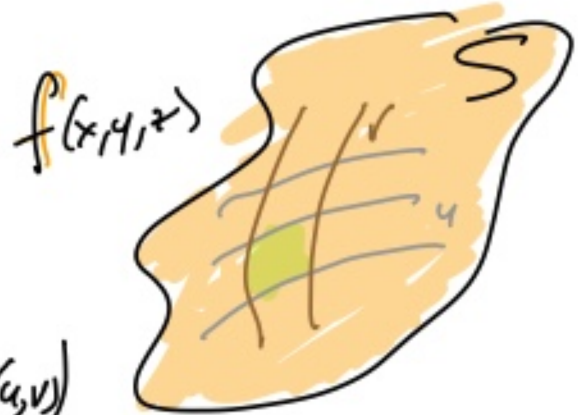
$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

$$\rho = 4$$

Parametrization $F(\theta,\phi) = (4\sin\phi\cos\theta, 4\sin\phi\sin\theta, 4\cos\phi)$

Integral of a function over a Surface:

$$\int_S f \, dA$$



Parametrize: $\Phi(u,v) = (x(u,v), y(u,v), z(u,v))$

$dA =$ area of

$du\,dv$

$$dA = |\Phi_u \times \Phi_v| \, du \, dv$$

Example: to calculate surface area:

$$\text{Surface Area} = \iint 1 \cdot |\Phi_u \times \Phi_v| \, du \, dv$$

In general $\displaystyle\int_S f \, dA = \iint f(x(u,v), y(u,v), z(u,v)) \, |\Phi_u \times \Phi_v| \, du \, dv$

↑ Integral of a scalar function over a surface.

Example Find the surface area of

$$z = x^3 y - y^2$$

for $0 \le x \le 2$
$-1 \le y \le 4$

$$\Phi(u,v) = (u, v, u^3 v - v^2)$$

$$\Phi_u = (1, 0, 3u^2 v)$$

$$\Phi_v = (0, 1, u^3 - 2v)$$

$$\Phi_u \times \Phi_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 3u^2 v \\ 0 & 1 & u^3 - 2v \end{vmatrix}$$

$$= -i(3u^2 v) - j(u^3 - 2v) + k(1)$$

$$= (-3u^2 v, -u^3 + 2v, 1)$$

$$dA = |\Phi_u \times \Phi_v| \, du\, dv$$

$$= \sqrt{9u^4 v^2 + (-u^3 + 2v)^2 + 1} \, du\, dv$$

Surface Area $= \int_{v=-1}^{4} \int_{u=0}^{2}$ ''   —